Recall:

First order linear ODE: $y' + p(t)y = g(t)$

Int. factor $\mu(t) = e^{\int p(t)dt}$

Gen. sol'n: $y(t) = \dfrac{\int \mu(t) g(t)\, dt + C}{\mu(t)}$

Separable ODE: $y' = f(x) g(y)$

Gen. sol'n: $\int \dfrac{dy}{g(y)} = \int f(x)dx + C$

## Exact ODE:

$$M(x,y) + N(x,y)y' = 0$$

satisfies

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{or} \quad M_y = N_x$$

Recall: Exact differential form: $M\,dx + N\,dy$

s.t. there exists a two-variable function $\psi(x,y)$, differentiable

$$d\psi(x,y) = M\,dx + N\,dy$$

or. $\dfrac{\partial \psi}{\partial x} = M$, $\dfrac{\partial \psi}{\partial y} = N$.

Notice for a differentiable function, $\dfrac{\partial^2 \psi}{\partial x \partial y} = \dfrac{\partial^2 \psi}{\partial y \partial x}$

$$\Rightarrow M_y = N_x$$

Conversely, if $M, N$ satisfies $M_y = N_x$, then there exists a function $\psi(x,y)$, s.t. $\frac{\partial \psi}{\partial x} = M$, $\frac{\partial \psi}{\partial y} = N$

From Calc 3, by the existence of $\psi(x,y)$, one can conclude that for an exact ODE, the general implicit solution is

$$\psi(x,y) = C$$

where $\frac{\partial \psi}{\partial x} = M$, $\frac{\partial \psi}{\partial y} = N$.

## How to recover this $\psi(x,y)$ ?

From $\frac{\partial \psi}{\partial x} = M(x,y)$. integrate both sides with the variable $x$

$$\psi(x,y) = \int M(x,y)\,dx + h(y)$$

$h(y)$ is to be determined by $\frac{\partial \psi}{\partial y} = N(x,y)$.

## General steps:

1. Check the exactness, i.e., if $M_y = N_x$
2. Integrate $M$ with respect to $x$ — or integrate $N$ w.r.t. $y$

   $\psi(x,y) = \int M(x,y)\,dx + h(y)$. — $\psi(x,y) = \int N\,dy + h(x)$
3. Use the condition $\frac{\partial \psi}{\partial y} = N(x,y)$ to recover $h(y)$ — or $\frac{\partial \psi}{\partial x} = M$ to get $h(x)$
4. Formulate the general sol'n $\psi(x,y) = C$

Example: $(3x^2y+2xy^5+5y)+(x^3+5x^2y^4+5x)\,y'=0$

Step 1: Check exactness:

$$M=3x^2y+2xy^5+5y\,,\quad N=x^3+5x^2y^4+5x$$

$$M_y=3x^2+10xy^4+5\,,\quad N_x=3x^2+10xy^4+5$$

exact ✓

Step 2: Integrate with one variable

Caution: should integrate $M$. not $M_y$ or $N_x$.

$$\psi(x,y)=\int M\,dx=\int(3x^2y+2xy^5+5y)\,dx$$

$$=x^3y+x^2y^5+5xy+h(y)$$

$\psi(x,y)=\int N\,dy+h(x)$

$\frac{\partial\psi}{\partial x}=M \Rightarrow h'(x)$

$\Rightarrow h(x)$

Step 3: Use $\frac{\partial\psi}{\partial y}=N$ to determine $h$.

$$\frac{\partial\psi}{\partial y}=x^3+5x^2y^4+5x+h'(y)$$

Set it equal to $N$

$$x^3+5x^2y^4+5x+h'(y)=x^3+5x^2y^4+5x$$

$$\Rightarrow h'(y)=0 \Rightarrow h(y)=0$$

we don't care about the arbitrary constant when integrating $h'(y)$

Step 4: Formulate the general implicit solution

$\psi(x,y)=x^3y+x^2y^5+5xy\cdot$ this is not the general sol'n!

General solution: $\psi(x,y)=C$

$$x^3y+x^2y^5+5xy=C$$

Remarks:

* If in your computation, $h'(y)$ is dependent of $x$, don't you dare to continue!
* Don't mistaken $M$ and $N$.

Example: $\underbrace{(3y^2e^{x^2+y^3}+x)}_{N}y'+\underbrace{2xe^{x^2+y^3}+y}_{M}=0$

$N$ is always the coefficient of $y'$ (or coeff. of $dy$)

$M$ is always the "constant" term (or coeff. of $dx$)

Example: $(3y^2e^{x^2+y^3}+x)y'+2xe^{x^2+y^3}+y=0$

Check exactness:

$$M=2xe^{x^2+y^3}+y.\quad N=3y^2e^{x^2+y^3}+x$$

$$M_y=6xy^2e^{x^2+y^3}+1,\quad N_x=6xy^2e^{x^2+y^3}+1$$

exact ✓

Integrate $N$ w.r.t. $y$.

$$\psi(x,y)=\int(3y^2e^{x^2+y^3}+x)\,dy=xy+\int 3y^2e^{x^2+y^3}\,dy$$

u-substitution: $u = x^2+y^3$. $du = 3y^2 dy$

$$\int 3y^2 e^{x^2+y^3} dy = \int e^u du = e^u + C$$

$$\psi(x,y) = xy + e^{x^2+y^3} + h(x)$$

$$\frac{\partial \psi}{\partial x} = M \Rightarrow y + 2xe^{x^2+y^3} + h'(x) = 2xe^{x^2+y^3} + y$$

$$\Rightarrow h'(x) = 0 \Rightarrow h(x) = 0$$

General solution:

$$xy + e^{x^2+y^3} = C$$

Example: $9x^2+y-1-(4y-x)y' = 0$, $y(1) = 0$

Find the explicit solution and determine the interval of existence.

Check exactness: $M = 9x^2+y-1$. $N = x-4y$

$M_y = 1$, $N_x = 1$, exact ✓

Integrate $M$ w.r.t. $x$.

$$\psi(x,y) = \int (9x^2+y-1)\,dx = 3x^3 + xy - x + h(y)$$

Determine $h(y)$

$$x + h'(y) = x - 4y \Rightarrow h'(y) = -4y \Rightarrow h(y) = -2y^2$$

General solution

$$3x^3+xy-x-2y^2=C$$

Initial value: $y(1)=0 \Rightarrow x=1,\ y=0$

$$\Rightarrow 3\times 1^3-0-1-2\times 0^2=C \Rightarrow C=2$$

$$3x^3+xy-x-2y^2=2$$

Explicit solution: Regard the above eqn as a quadratic equation concerning $y$, regard all $x$'s as constants.

$$2y^2-xy-3x^3+x+2=0$$

Quadratic formula

$$\Rightarrow y=\frac{+x\pm\sqrt{x^2-4\times2\times(-3x^3+x+2)}}{2\times2}$$

$$=\frac{1}{4}\left(x\pm\sqrt{24x^3+x^2-8x-16}\right)$$

$$y(1)=0,\ \text{put in } x=1 \Rightarrow y=\frac{1}{4}\left(1\pm\sqrt{\cancel{24}+1-\cancel{8}-\cancel{16}}\right)$$

$$=\frac{1}{4}(1\pm1)$$

b/c $y(1)=0$, we should choose negative branch.

Explicit solution:

$$y=\frac{1}{4}\left(x-\sqrt{24x^3+x^2-8x-16}\right)$$

Use maple or other softwares to solve

$$24x^3 + x^2 - 8x - 16 \geq 0$$

$$\Rightarrow \quad x \geq 0.9846$$

Interval of existence $[0.9846, \infty)$.

Attendance Quiz: Find the general solution to

$$(3x^2y + 2xy^5 + 7y) + (x^3 + 5x^2y^4 + 7x + 2\sin 2y)y' = 0.$$

## What can be done if the ODE is not exact?

Sadly, there's no way to deal with all possible cases

However, under certain conditions, the ODE can be made into an exact ODE

## Method of Integrating factor.

Idea: Multiply $\mu(x,y)$ to the ODE

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0$$

Want this ODE to be exact.

It's not always possible to find such an integrating factor

In fact, this ODE being exact means

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

$$\frac{\partial \mu}{\partial y} M + \mu \cdot \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x} N + \mu \cdot \frac{\partial N}{\partial x}.$$

In general we cannot recover from this equation.

However. if $\mu(x,y)$ is independent of $y$, i.e., $\mu = \mu(x)$. $\frac{\partial \mu}{\partial y} = 0$

$$0 \cdot M + \mu \cdot \frac{\partial M}{\partial y} = \frac{d\mu}{dx} N + \mu \cdot \frac{\partial N}{\partial x}$$

$$\frac{d\mu}{dx} \cdot N = \mu (M_y - N_x)$$

$$\frac{d\mu}{\mu} = \frac{M_y - N_x}{N} dx$$

LHS being a function concerning $x$ only.

RHS should also depend only on $x$

If $\frac{M_y - N_x}{N}$ is independent of $y$, i.e., it depends only on $x$.

then by solving $\frac{d\mu}{\mu} = \frac{M_y - N_x}{N} dx$

one can obtain an integrating factor.

Similarly, if $\mu(x,y)$ is independent of $x$, a similar argument gives:

If $\frac{M_y - N_x}{-M}$ is independent of $x$, i.e., it depends only on $y$,

then by solving $\frac{d\mu}{\mu} = \frac{M_y - N_x}{-M} dy$

one obtains an integrating factor.

Example: $3xy^2+2y^3+(2x^2y+3xy^2)y'=0$

Check exactness:

$$M=3xy^2+2y^3, \quad N=2x^2y+3xy^2$$

$$M_y=6xy+6y^2, \quad N_x=4xy+3y^2$$

$$M_y-N_x=2xy+3y^2 \qquad \text{not exact}$$

$$\frac{M_y-N_x}{N}=\frac{2xy+3y^2}{2x^2y+3xy^2}=\frac{2xy+3y^2}{x(2xy+3y^2)}=\frac{1}{x} \quad \text{indep. of } y$$

Integrating factor:

$$\frac{d\mu}{\mu}=\frac{1}{x}dx \Rightarrow \ln|\mu|=\ln|x| \Rightarrow \mu=x$$

$$\ln|\mu|=\ln|x|+c \Rightarrow \mu=Cx$$

New ODE: $3x^2y^2+2xy^3+(2x^3y+3x^2y^2)y'=0$

$$3Cx^2y^2+2Cxy^3+(2Cx^3y+3Cx^2y^2)y'=0$$

$$M=3x^2y^2+2xy^3, \quad N=2x^3y+3x^2y^2$$

$$M_y=3x^2\cdot 2y+6xy^2, \quad N_x=6x^2y+6xy^2 \quad \text{exact} \checkmark$$

Continue to solve it as an exact ODE.

$$\psi(x,y)=\int(3x^2y^2+2xy^3)\,dx=x^3y^2+x^2y^3+h(y)$$

$$\frac{\partial \psi}{\partial y} = 2x^3y + 3x^2y^2 + h'(y) = N = 2x^3y + 3x^2y^2$$

$$\Rightarrow h'(y) = 0 \Rightarrow h(y) = 0$$

General solution:

$$x^3y^2 + x^2y^3 = C.$$

# Lecture Notes of Differential Equation

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Lecture Notes of Differential Equation